Kuga-Satake varieties of a family of K3 surfaces of Picard rank 14

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Introduction

A Kuga-Satake (KS) variety is an abelian variety associated to a K3 surface. We would like to study this correspondence on the moduli level for a special family of K3 surfaces.

Rundown

1. Recall definitions: K3 surfaces, abelian varieties and their Torelli theorems
2. The KS construction
3. Some possible questions to study
4. Why the family of K3 surface of Picard rank 14

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Definitions

K3 surfaces and abelian varieties

K3 surface X:
Simply connected, compact, complex 2-fold with trivial canonical bundle

Polarisation by lattice M:
$j\colon M \hookrightarrow \mathrm{Pic}(X)$

Abelian variety A of dimension g :
Complex torus with a polarisation $(\mathbb{C}^g/\Lambda, h)$
where Λ: full rank lattice in $\mathbb{C}^g$
polarisation h: hermitian form on $\mathbb{C}^g$ with a +-ve def. imaginary part

Note: upon choosing a suitable basis, the matrix of h can be brought to

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, , d_g)$ with $d_i | d_{i+1}$ for all $1 \leq i \leq g-1$
We say $(d_1, \cdots, d_g)$ is the polarisation type of A

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Hodge structure and Torelli theorem

Hodge structure (HS) of weight n:

V: free $\mathbb{Z}$-module of finite rank or $\mathbb{Q}$-vector space of finite dimension

A HS of weight n on V is the decomposition of $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

such that $V^{p,q} = \overline{V^{q,p}}$.

Example of HS:

Each cohomology group of a Kähler manifold X has a HS coming from Dolbeaut cohomology

$$H^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X) := H^q(X, \Omega^p)$.

Hodge structure and Torelli theorem

e.g. HS of cohomology groups of K3 surfaces

Hodge diamond of a K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Torelli theorem for K3 surfaces:
Fix lattice M. An M-polarised K3 surface X is uniquely determined by the weight two HS on $H^2(X)$ up to isomorphism.

$\Rightarrow$ In a moduli space of K3 surfaces polarised by a fixed lattice M, the weight two HS of second cohomology group ($H^{2,0}$) parametrises the K3 surfaces.

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For a complex torus $T \simeq V/\Lambda$,

$$T \simeq {}^{H^{0,1}(T)^*}\big/_{\left(H^1(T,\mathbb{Z})^* \cap H^{0,1}(T)^*\right)}$$

Torelli theorem for abelian varieties:

Fix a polarisation type for abelian g-folds. An abelian variety A_g with the polarisation type is uniquely determined by the weight one HS on $H^1(A_g)$ up to isomorphism.

$\Rightarrow$ In a moduli space of abelian varieties of a fixed polarisation type, the weight one HS of first cohomology group (complex structure) parametrises the abelian varieties.

What if we can create a weight one HS from a weight two HS...

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KS construction

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[Kuga, Satake, 1967]

$\big(X, j : M \hookrightarrow \mathrm{Pic}(X)\big)$: K3 surface polarised by lattice M

$V := M^{\perp}_{H^2(X,\mathbb{Z})}$ with quadratic form q

Consider the Clifford algebra generated from V

$$Cl(V) = {\bigotimes V}\big/{I(q)}$$

where $\otimes V$: tensor algebra generated from V
$I(q)$: ideal generated by $v \otimes v - q(v)$ for all $v \in V$

Note: $Cl(V)$ inherits grading from $\otimes V$.
We denote $Cl^{+}(V)$ as the even part of $Cl(V)$.

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KS construction

Upon choosing a complex structure for the real extension of the even part $Cl^+(V_\mathbb{R})$, we can construct a torus

$$KS(V) := {}^{Cl^+(V)^{1,0}}\big/_{Cl^+(V) \cap Cl^+(V)^{1,0}}$$

which comes with a polarisation.

$\Rightarrow$ the KS variety $KS(V)$ constructed from the K3 surface X is indeed an abelian variety!

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Questions to study

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Consider any K3 surfaces X polarised by the rank 14 lattice

$$M = U \oplus D_8(-1) \oplus D_4(-1)$$

The complement of M in $H^2(X)$ is

$$V = U \oplus U(-2) \oplus D_4(-1)$$

which has signature $(2, 6)$.

$KS(V)$ is an abelian 64-fold,
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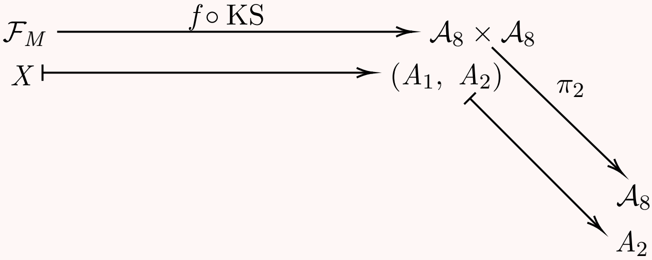
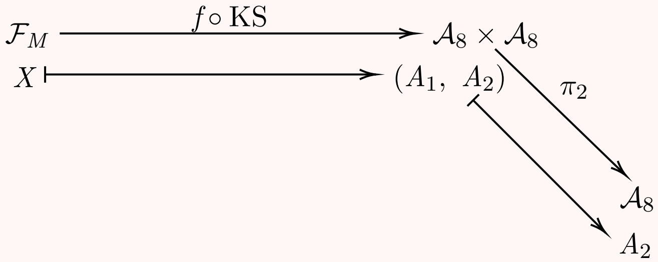
Upon choosing a suitable isogeny $f\colon \mathrm{KS}(V) \to (A_1, A_2)$ "globally", we have on the moduli level:



where $\mathcal{F}_M$: moduli space of K3 surfaces polarised by M
$\mathcal{A}_8$: moduli space of abelian 8-folds with certain polarisation type and endomorphism structure
π_2: projection map.

Questions to study



Fact: Both $\mathcal{F}_M$ and $\mathcal{A}_8$ are of dimension 6!

Some questions to ask:

- Is $\pi_2 \circ f \circ \mathrm{KS}$ dominant?
- What is the degree of π_2?
- What does degeneration in $\mathcal{F}_M$ correspond to in $\mathcal{A}_8$?
- What are the answers to these questions if we start with a different family of K3 surfaces of Picard rank 14?

Why rank 14? Why this M?

Why rank 14?

- Moduli spaces of K3 surfaces polarised by a rank r lattice sometimes coincide with other moduli spaces.

r	Moduli space of (polarised)
17	abelian surfaces
16	generalised Kummer varieties
15	OG10

We don't know about $r = 14$...

- Gives insight of the correspondence of locally symmetric spaces

$$II_4 \sim IV_6$$

Moduli spaces of K3 surfaces of Picard rank 14

$$II_4 \hookrightarrow III_8$$

Moduli spaces of polarised abelian 8-folds

suggested by the exceptional isomorphism of lie algebras

$$so^*(8) \simeq so^+(2,6)$$

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Why M?

- [Clingher, Malmendier, 2022]
  Studied 3 families of K3 surfaces of Picard rank 14 with significance in String theory. One of which is our $\mathcal{F}_M$.

Thank you!